Graphs and Their Applications (8)

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23 Bipartite graphs

We begin with a discussion of the following problem:

Five applicants A, B, C, D, E apply to work in a company. There are six jobs available: $J_1, \cdots, J_6$. Applicant A is qualified for jobs J_2 and J_6; B is qualified for jobs J_1, J_3 and J_4; C is qualified for jobs J_2, J_3 and J_6; D is qualified for jobs J_1, J_2 and J_3; E is qualified for all jobs except J_4 and J_6. Is it possible to assign each applicant to a job for which he/she is qualified?

It is really not quite obvious to see if there is any solution to this problem from the wording. Let us use a graph to model the situation. Figure 23.1 shows a graph with 11 vertices representing 5 applicants and 6 jobs in which two vertices are joined by an edge if the corresponding applicant is qualified for the corresponding job.

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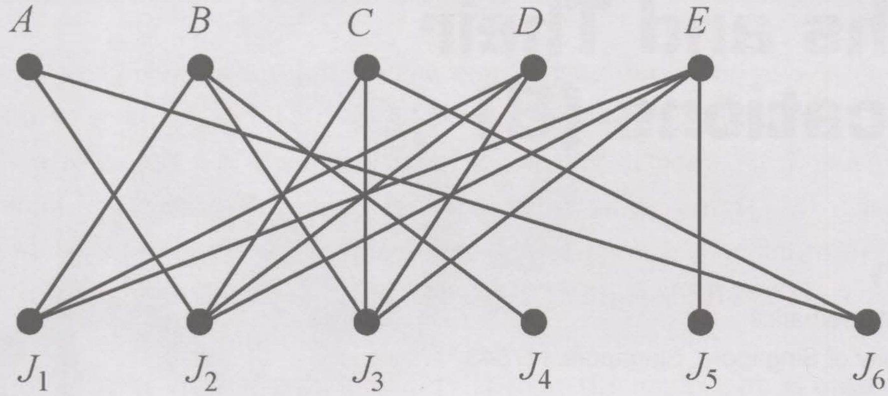
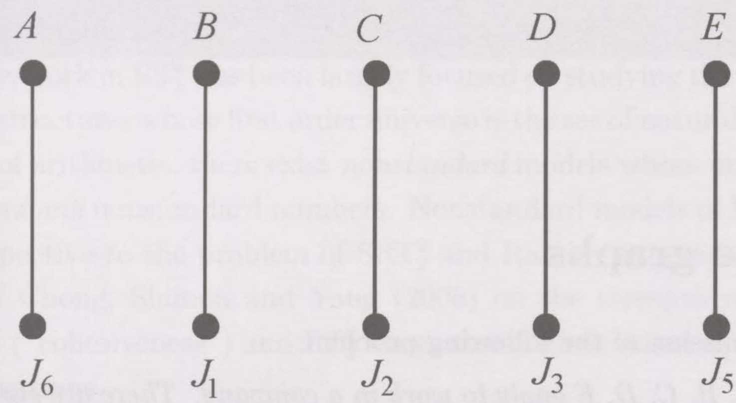


Figure 23.1

With the help of the graph, it is now not too difficult to see that it is possible to assign each applicant to a job for which he/she is qualified. One assignment is as follows:



At this initial stage of our study, our main concern actually is not to find such an assignment but to investigate the 'structure' of the graph shown in Figure 23.1. Is there any special feature of this graph?

We notice that the 11 vertices of the graph are *divided into two parts* (upper and lower) in such a way that *any edge in the graph joins a vertex in one part to a vertex in another part; that is, no two vertices in the same part are joined by an edge.*

Graphs with this feature are very important and useful. We call them **bipartite graphs**.

A graph G is said to be **bipartite** if its vertex set $V(G)$ can be divided into two disjoint subsets, say X and Y, such that each of the edges in G joins a vertex in X to a vertex in Y. In this case, we call X and Y the **partite sets** of G, and call the pair (X, Y) a **bipartition** of G.

Thus, if G is a bipartite graph with bipartition (X, Y), then no two vertices in X are joined by an edge (the same is true for Y).

Exercise 23.1 *A bipartite graph G with bipartition (X, Y) is defined as follows: $X = \{a, b, c\}$ and $Y = \{u, v, w\}$, and $E(G) = \{aw, bu, bw, cu, cv\}$. Draw the graph G.*

Exercise 23.2 *A graph G is defined as follows: $V(G) = X \cup Y$, where*

$$X = \{2, 3, 5\} \quad \text{and} \quad Y = \{5, 10, 15, 20\},$$

and $E(G) = \{xy | x \in X, y \in Y$ and y is divisible by $x\}$.

(i) Is '2' in X adjacent to '5' in Y? Is '2' in X adjacent to '10' in Y? Is '3' in X adjacent to '15' in Y?

(ii) Draw the graph G.

(iii) Is G bipartite?

Now, consider the graph H of Figure 23.2. We ask: is it a bipartite graph?

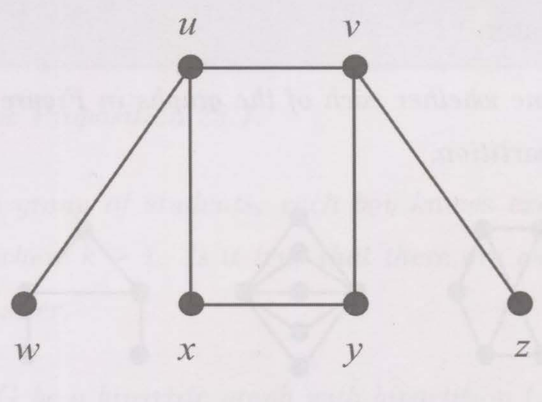


Figure 23.2

To answer this question, we may first ask ourselves (recalling the definition): Does H possess a bipartition (X, Y)?

If we take $A = \{u, v\}$ and $B = \{w, x, y, z\}$, is (A, B) a bipartition of H?

The answer is 'NO'. Why? Because no edge is allowed to join two vertices in the same partite set (in this case, A or B).

Though, in this case, (A, B) is not a bipartition of H, it does not mean that H is not bipartite. Indeed, if we take $X = \{u, y, z\}$ and $Y = \{w, x, v\}$, then we observe that

(i) $V(H) = X \cup Y$ and

(ii) each edge in H joins a vertex in X to a vertex in Y.

It follows by definition that (X, Y) is a bipartition of H, and thus H is, in fact, a bipartite graph.

Let us re-draw the graph H in a more 'natural' form as shown in Figure 23.3. It is now clear from the diagram that H is a bipartite graph.

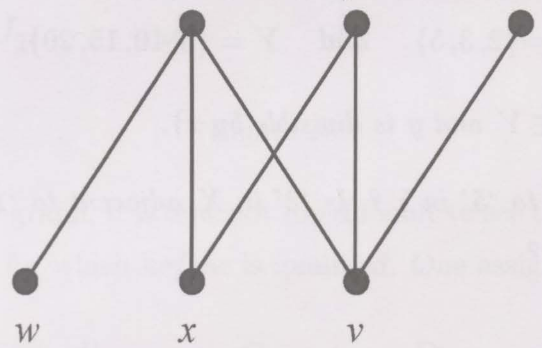


Figure 23.3

Quite often, bipartite graphs are not drawn in this 'natural' form. How to find a bipartition of a bipartite graph thus becomes a practical and interesting problem. We shall discuss this problem later.

Exercise 23.3 *Determine whether each of the graphs in Figure 23.4 is bipartite. If it is, find a corresponding bipartition.*

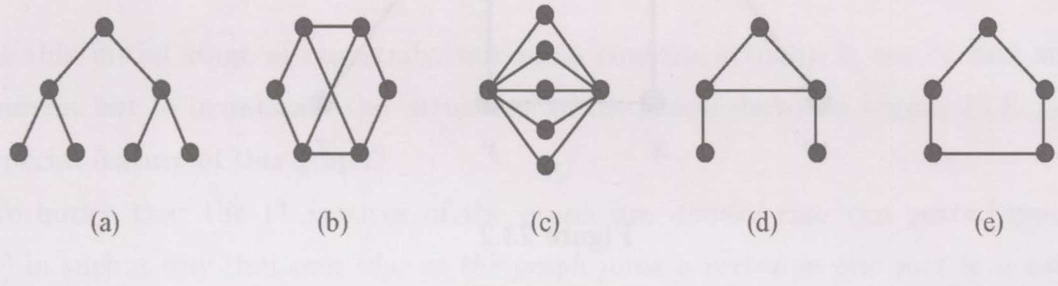


Figure 23.4

Recall that Euler's handshaking lemma (see [1]) states that for any multigraph G,

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

Suppose now that G is a bipartite graph with bipartition (X, Y). Then $\sum\limits_{v \in V(G)} d(v)$ can be split naturally into two parts, namely,

$$\sum_{v \in X} d(v) \text{ and } \sum_{v \in Y} d(v).$$

What can we say about these two sums?

Take, for instance, the bipartite graph G of Figure 23.1 with bipartition (X, Y), where $X = \{A, B, C, D, E\}$ and $Y = \{J_1, J_2, \cdots, J_6\}$. Observe that

$$\sum_{v \in X} d(v) = d(A) + d(B) + \cdots + d(E) = 2 + 3 + 3 + 3 + 4 = 15 \text{ and}$$

$$\sum_{v \in Y} d(v) = d(J_1) + d(J_2) + \cdots + d(J_6) = 3 + 4 + 4 + 1 + 1 + 2 = 15;$$

that is, they are both equal to '15'.

How many edges are there in G? Are there any relationships among $\sum_{v \in X} d(v)$, $\sum_{v \in Y} d(v)$ and $e(G)$?

Indeed, we have the following simple but useful result, which is a refinement of the handshaking lemma for bipartite graphs.

Proposition 23.1

Let G be a bipartite graph with bipartition (X, Y). Then

$$\sum_{x \in X} d(x) = e(G) = \sum_{y \in Y} d(y).$$

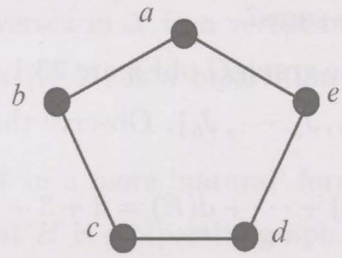
Exercise 23.4 *Prove Proposition 23.1.*

Exercise 23.5 *In a group of students, each boy knows exactly k girls and each girl knows exactly k boys, where $k \geq 1$. Is it true that there are as many girls as boys in the group? Justify your answer.*

Exercise 23.6 *Let G be a bipartite graph with bipartition (X, Y). Assume that G has the same order and size, and that $d(x) \leq 7$ for each $x \in X$. Prove that $|Y| \leq 6|X|$.*

24 König's characterization

You might have solved Exercise 23.3 and found out, in Figure 23.4, that the graphs (a), (b) and (c) are bipartite, whereas (d) and (e) are not. Let us examine the graph (e), which is C_5, the cycle of order 5. For convenience, we name its five vertices as shown below:



Suppose on the contrary that it is bipartite and has a bipartition (X, Y). We may assume that $a \in X$. As no two adjacent vertices can be in the same partite set, we must have $b \in Y$. This, in turn, implies (anti-clockwise) that $c \in X$, $d \in Y$ and $e \in X$. We thus arrive at the situation that both a and e are in X and they are adjacent, which, however, is not allowed. This shows that C_5 possesses no bipartition, and is therefore not bipartite.

A cycle C_k is said to be **odd** (resp., **even**) if k is odd (resp., even).

From the above discussion, we actually see that the argument can similarly be carried out to lead to a contradiction as long as there is an odd cycle. Thus, we conclude that

if G contains an odd cycle, then G is not bipartite;

or equivalently,

if G is bipartite, then G contains no odd cycles.

Is the converse of this result true? That is, if G contains no odd cycles, must G be bipartite?

Exercise 24.1 *The graphs (a), (b) and (c) in Figure 23.4 contain no odd cycles and they are bipartite. Consider the graph of Figure 24.1.*

(i) Does it contain any odd cycle?

(ii) Is it bipartite?

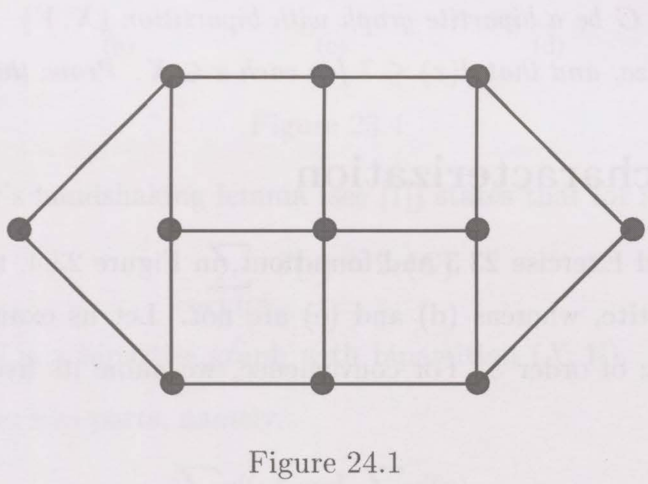


Figure 24.1

Yes! If a graph contains no odd cycles, then it must be bipartite! This result was found by the Hungarian combinatorialist Denes König (1884-1944) in 1916, who wrote the first book [2] on graph theory in 1936.

D. KÖNIG (1884–1944)

Theorem 24.1

A graph G is bipartite if and only if it contains no odd cycles.

The proof of the necessity, that is, 'if G is bipartite, then G contains no odd cycles', can be carried out as how we did above, and is left to the reader (see Exercise 24.2).

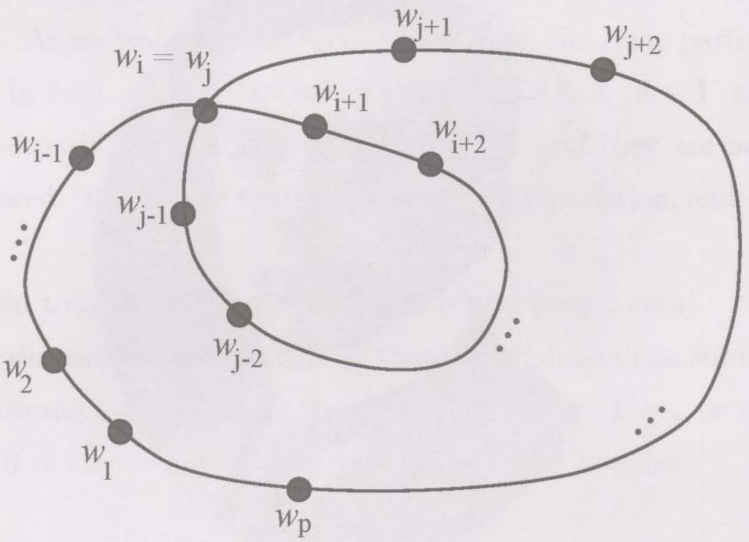
We shall now prove the sufficiency, that is, 'if G contains no odd cycles, then G is bipartite'. For this purpose, we first prove the following simple observation:

Lemma 24.2 *Every closed walk of odd length in a graph always contains an odd cycle.*

Proof. Let W be a closed walk of odd length $p \geq 3$. We shall prove the statement by induction on p.

For $p = 3$, we have $W = w_1w_2w_3w_1$, which obviously forms a C_3.

Assume that it is true for all closed walks of odd length less than p, where $p \geq 5$. Now consider a closed walk of odd length $p : W = w_1w_2 \cdots w_pw_1$. Our aim is to show that W contains an odd cycle. If W forms itself a C_p, we are through; otherwise, some vertices are repeated and there exist i and j with $1 \leq i < j \leq p$ such that $w_i = w_j$, as shown bwlow:



Consider the two closed walks of smaller length:

$$W_1 = w_1w_2 \cdots w_iw_{j+1} \cdots w_pw_1$$

and

$$W_2 = w_iw_{i+1} \cdots w_{j-1}w_j \qquad \text{(note that } w_j = w_i\text{)}.$$

One of them must be of odd length (why?), say W_1 . As the (odd) length of W_1 is less than p, by the induction hypothesis, W_1 contains an odd cycle. Clearly, this odd cycle is contained in W. The proof is thus complete.

□

With the help of Lemma 24.2, we are now ready to prove the sufficiency of Theorem 24.1.

Proof. We assume that G contains no odd cycles, and aim to show that G is bipartite by providing a bipartition of G.

We may assume that G is connected (why?). Let w be a fixed vertex in G and let

$$X = \{v \in V(G) | d(w, v) \text{ is even}\}$$

and $Y = \{v \in V(G) | d(w, v) \text{ is odd}\}$.

We now claim that (X, Y) **is a bipartition of** G.

It is obvious that X and Y are disjoint, and as G is connected, $V(G) = X \cup Y$ (note that $w \in X$).

It remains to show that each edge in G joins a vertex in X to a vertex in Y. Suppose this is not the case. Then there exist u, v in X or u, v in Y, say the former, such that $uv \in E(G)$. As u, v are in X, by definition, $d(w, u)$ and $d(w, v)$ are even, and there exist a $w - u$ path P of even length and a $v - w$ path Q of even length in G.

Consider the walk W which begins at w, follows P to reach u, passes 'uv' to reach v, and finally follows Q to return to w. This walk W is closed and of odd length (why?). By Lemma 24.2, W contains an odd cycle. But then this implies that G contains an odd cycle, a contradiction.

The proof is thus complete. □

Note. The argument given in the above proof suggests a way to find a bipartition of a connected graph G if G contains no odd cycles. The procedure is as follows:

(1) Begin with an (arbitrary) vertex and label it '1'.

(2) Suppose a vertex has been labeled '1', label all its neighbours '2'; and if a vertex has been labeled '2', label all its neighbours '1'.

(3) Repeat (2) until all vertices have been labeled.

Then the set of all vertices with label '1' and the set of vertices with label '2' form a bipartition of G.

Exercise 24.2 *Let G be a graph. Show that if G is bipartite, then G contains no odd cycles.*

Exercise 24.3 *Apply the procedure stated in the above note to find a bipartition for each of the graphs in Figure 23.2, Figure 23.4(a), (b) and (c), and Figure 24.1.*

Exercise 24.4 *The graph of Figure 24.2 is not bipartite. Apply the procedure stated in the above note to identify an odd cycle.*

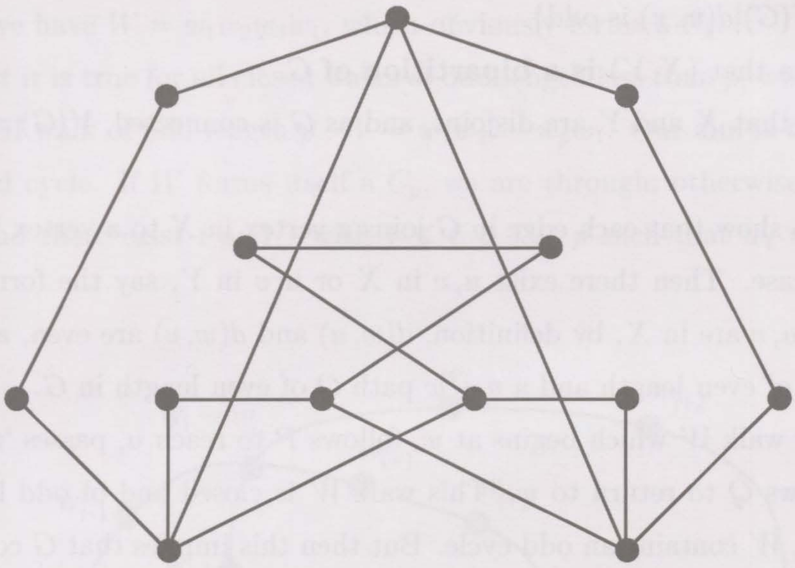


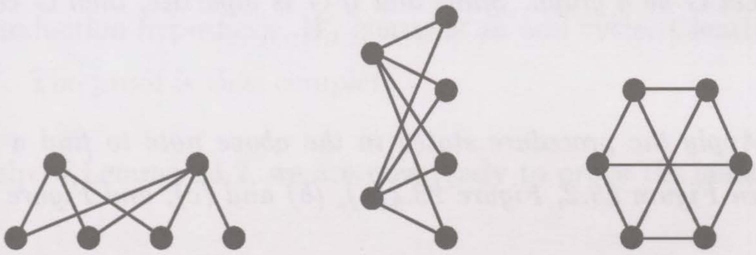
Figure 24.2

25 Complete bipartite graphs

By definition, a bipartite graph possesses a bipartition (X, Y) such that each edge in the graph joins a vertex in X to a vertex in Y. Note that we do not require that every vertex in X must be adjacent to every vertex in Y. Indeed, this extreme case happens only in a special family of bipartite graphs.

> A bipartite graph with bipartition (X, Y) is called a **complete bipartite graph** if each vertex in X is adjacent to each vertex in Y.

Exercise 25.1 *Which of the following bipartite graphs is a complete bipartite graph?*



(a) (b) (c)

There is one and only one (up to isomorphism) *complete* bipartite graph with a given bipartition (X, Y). If $|X| = p$ and $|Y| = q$, we shall denote this complete bipartite graph by $K(p,q)$ or $K_{p,q}$. Note that the graphs $K(1,q)$ and $K(p,1)$ are trees, and we call such graphs **stars** (see Figure 25.1).

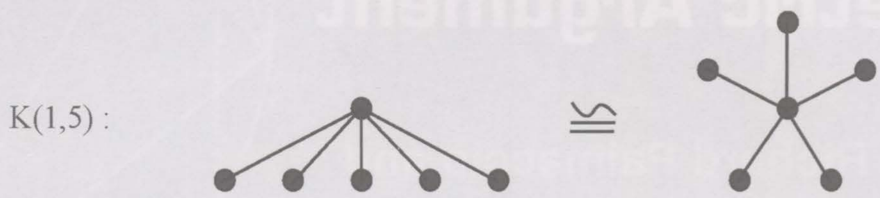


Figure 25.1

Exercise 25.2 *(i) Draw $K(3,5)$ and $K(5,3)$.*

(ii) Is $K(3,5)$ isomorphic to $K(5,3)$?

(iii) Find $e(K(3,5))$.

(iv) Find the degree of each vertex in $K(3,5)$.

We end this section by stating the following result.

Proposition 25.1 *For all positive integers p and q,*

(1) $K(p,q) \cong K(q,p)$;

(2) $e(K(p,q)) = pq$; and

(3) in $K(p,q)$, the vertices in X are of degree q while those in Y are of degree p.

References

[1] K.M. Koh, Graphs and their application (1), Mathematical Medley, 29(2) (2002), 86-94

[2] D. König, *Theorie der endlichen und unendlichen Graphen*, Akademische Verlagsgesellschaft, Leipzig, 1936